Improper Integral

part II

$$\int_{x=a}^{x=b} f(x)\,dx \quad \text{but either} \quad a,b \in \mathbb{R}$$

① $f(x)$ is not Continuous at $x=a$

or ② $f(x)$ is not Continuous at $x=b$

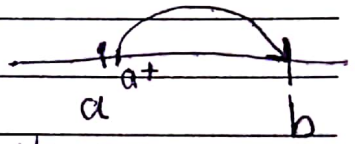
or ③ $f(x)$ is not Continuous at $x=c$; $c \in (a,b)$

a : صفر للمقام

① If $f(x)$ is not Continuous at $x=a$

$$\int_{x=a}^{x=b} f(x)\,dx \Rightarrow \lim_{c \to a^+} \int_c^b f(x)\,dx$$

If limit exist $\Rightarrow \int_a^b f(x)\,dx$ Converge



If limit doesn't exist

$\Rightarrow \int_a^b f(x)\,dx$ diverges

② If $f(x)$ is not Continuous at $x=b$

b : صفر للمقام

$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

limit exists $\Rightarrow \int$: Converge

limit d.n.e $\Rightarrow \int$: Diverge

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③ If $f(x)$ is not continuous at $c \in (a,b)$

$\int_a^b f(x)\,dx$: $f(x)$ is not Cont. at $x = c$

$\Rightarrow$ we should divide the Integral over the interval (a,b) as follows

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{z \to c^-} \int_a^z f(x)\,dx + \lim_{z \to c^+} \int_z^b f(x)\,dx$$

If both limits exists $\Rightarrow \int_a^b f(x)\,dx$ Converge

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Ex:① Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1-\tan x}\,dx$

note: when $x=\frac{\pi}{4} \Rightarrow = 1-\tan\frac{\pi}{4} =$

$= 1-1=0 \Rightarrow$ denominator will be zero $\Rightarrow$ improper integral

$$\lim_{c\to \frac{\pi}{4}^-} \int_0^c \frac{\sec^2 x}{1-\tan x}\,dx$$

$$\int \frac{\sec^2 x}{1-\tan x}\,dx$$

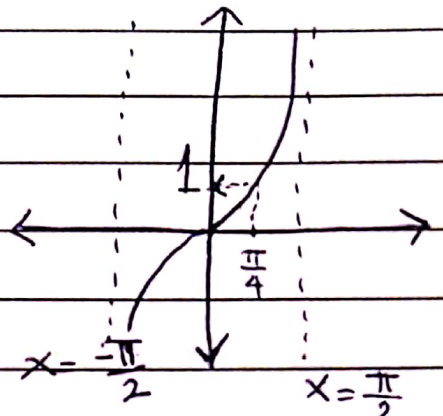
$z = \tan x$

$dz = \sec^2 x\,dx \Rightarrow \int \frac{dz}{1-z}$

$= \ln|1-\tan x|$

$$\lim_{c\to \frac{\pi}{4}^-} \left(\ln|1-\tan c| - \ln|1-\tan 0|\right)$$

but $\tan\frac{\pi}{4}^- = 1^-$



but $\ln 1 = 0$

$$\left(\ln|1-1^-| - \ln 1\right)$$

$= \ln 0^+ = -\infty \Rightarrow \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1-\tan x}\,dx$: diverges

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H.W: $\int_0^{\pi/2} \tan x \, dx$

Ex: (2) $\int_{-3}^{1} \frac{x \, dx}{\sqrt{9-x^2}}$ note: at $x=-3$

denominator: $\sqrt{9-(-3)^2} = 0$

improper since at $x=-3 \Rightarrow$ denominator $=0$

$$\int_{-3}^{1} \frac{x \, dx}{\sqrt{9-x^2}} = \lim_{c \to -3^+} \int_{c}^{1} \frac{x \, dx}{\sqrt{9-x^2}}$$

but $\int \frac{x}{\sqrt{9-x^2}} dx = \int \frac{du}{2\sqrt{9-u}}$ let $u = x^2$, $du = 2x \, dx$

$$= \frac{1}{2}\int (9-u)^{-1/2} du = \frac{1}{2} \frac{(9-u)^{1/2}}{-\frac{1}{2}} = -\sqrt{9-x^2}$$

$$\Rightarrow \lim_{c \to -3^+} \left[-\sqrt{9-(1)^2} - \left(-\sqrt{9-c^2}\right) \right]_{c \to -3^+}$$

$$= \left[-\sqrt{8} + \sqrt{0} \right] = -\sqrt{8} \Rightarrow \in \mathbb{R}$$

$\int_{-3}^{1} \frac{x \, dx}{\sqrt{9-x^2}}$ Converges

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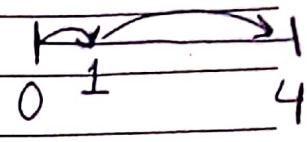
(Ex) $\int_0^4 \frac{dx}{(x-1)^{2/3}}$

note

$(x-1)^{2/3} = 0 \Rightarrow$

$x = 1$



$$\Rightarrow \int_0^4 = \int_0^1 + \int_1^4$$

- سبب إظهار المشكلة على أطراف التكامل

$$\int_0^4 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} + \lim_{c\to 1^+} \int_c^4 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^-} \left[3(c-1)^{1/3} - 3(0-1)^{1/3} \right] +$$

$$\lim_{c\to 1^+} \left[3(4-1)^{1/3} - 3(c-1)^{1/3} \right]$$

$$= (-3)(-1)^{1/3} + 3(3)^{1/3}$$

$$= +3 + 3(3)^{1/3} = 3(1+3^{1/3}) \in \mathbb{R}$$

$$\Rightarrow \int_0^4 \frac{dx}{(x-1)^{2/3}} \text{ Converg}$$

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Ex: determine the value(s) of "p" for which the integral $\int_0^1 \frac{dx}{x^p}$ is improper

Solu:

$$\int_0^1 x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_{x=0}^{x=1}$$

$$\frac{1}{-p+1}\left[(1)^{-p+1} - (0)^{-p+1}\right]$$

$$\frac{1}{-p+1}\left[1-0\right] = \frac{1}{-p+1}$$

Now:

if $-p+1=0 \Rightarrow p=1$

If $p=1 \Rightarrow \int_0^1 \frac{dx}{x}$ improper

If $p>1 \Rightarrow \int_0^1 \frac{dx}{x^3}$ not improper (for example $p=3$)

If $p<1 \Rightarrow \int_0^1 \frac{dx}{x^{\frac{1}{2}}}$ not improper ($p=\frac{1}{2}$) for example

Correct answer

$p \geq 1$

$p \in [1, \infty)$

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